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THE STRUCTURE OF ADMISSIBLE POINTS WITH RESPECT TO CONE DOMINANCE[+]

by

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The Structure of Admissible Points with Respect to Cone
Dominance

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Abstract

We study the set of admissible (pareto-optimal) points of a closed
convex set X when preferences are described by a convex, but not neces-
sarily closed, cone. Assuming that the preference cone is strictly sup-
ported and making mild assumptions about the recession directions of X,
we

(i) extend a representation theorem of Arrow, Barankin and Blackwell
by showing that all admissible points are either limit points of certain
"strictly admissible" alternatives or translations of such limit points
by rays in the closure of the preference cone, and

(ii) show that the set of strictly admissible points is connected,
as is the full set of admissible points.

Relaxing the convexity assumption imposed upon X, we also consider
local properties of admissible points in terms of Kuhn-Tucker type
characterizations. We specify necessary and sufficient conditions for
an element of X to be a Kuhn-Tucker point, conditions which, in addition,
provide local characterizations of strictly admissible points.

1. INTRODUCTION

Formal approaches to decision making almost always pre-
sume that an underlying preference relation governs choices
from available alternatives. Rich theories now go far toward
either characterizing or computing solutions that are in some
sense "optimal". Mathematical programming techniques predomi-
nate when preferences can be embodied in a real valued
utility function (Debreu [1], [2] discusses appropriate con-
ditions. See also Bowen [3] and Arrow and Hahn [4, page 106]).
Multi-attribute utility theory (Keeney and Raiffa [5]) pro-
vides one means for considering multi-objective situations
which involve several, possibly conflicting, criteria. In
summarizing methods for studying multi-objective decision
making, Mac Crimmon [6] has classified approaches as weight-
ing methods, including statistical analysis; sequential elim-
ination techniques; mathematical programming procedures;
and special proximity methods.

Although these theories have made impressive contribu-
tions to decision making, they have yet to resolve a number
of issues that are fundamental to both descriptive and
prescriptive theory. For arbitrary preference relations,
still little is known, and possibly can be said, about such
an essential concept as admissible alternatives, also called

nondominated, efficient, or pareto-optimal alternatives. Even when a convex cone $\mathbf{P}$, the set of points $x + \mathbf{P} \equiv \{x + p : p \in \mathbf{P}\}$ specifying those alternatives preferred to x, describes preferences, admissible points have not been characterized completely. Is the set of admissible points connected ? Are there representation theorems which characterize admissible points ? What are local characterizations of admissible points ? Can the notion of admissible points be exploited within the context of solving mathematical programs ?

In this paper, we consider several of these issues. In Section 2, we introduce notation and concepts to be used throughout the paper. The next section considers global characterizations of admissible points, including existence. We show that the sets of admissible and strictly admissible points are both connected when (i) the preference cone $\mathbf{P}$ is convex, (ii) the set $\mathbf{P}_s^+ \equiv \{p^+ \in \mathbf{R}^n : p^+ \cdot p > 0 \text{ for all nonzero } p \in \mathbf{P}\}$ is nonempty, (iii) the set of available alternatives X is convex and closed, and (iv) some element of $\mathbf{P}_s^+$ makes an obtuse angle with every direction of recession of X. In this section, we also present a representation theorem which partially characterizes admissible points. This result says that "all admissible points can be expressed as limit points of 'strictly' admissible alternatives or as a translation of such limit points by certain rays in the

closure of the preference cone $\mathbf{P}''$. Section 4 considers local characterization of admissible points in terms of linear approximations. These results are related to the usual Kuhn-Tucker characterizations of nonlinear programming. The final section discusses possible extensions and applications.

Our analysis is based upon results of convex analysis and mathematical programming. This approach is an outgrowth of work conducted around 1950 in mathematical statistics (Wald [7], Arrow, Barankin and Blackwell [8]), in linear and nonlinear programming (Gale, Kuhn and Tucker [9], Kuhn and Tucker [10]), and in economic planning (Koopmans [11]). These fundamental contributions either proved or suggested many of the properties that we consider here under various restrictions on the problem structure, most notably that the set of available alternatives is polyhedral and/or that preference $x \succ y$, x is preferred to y, is defined by $x \geqslant y$, $x \neq y$. Later in a series of papers [12], [13], and [14], Geoffrion studied properties and computational aspects of certain nonlinear vector maximization problems. More recently Yu [15] considered preferences defined by cones as here, including several results related to this paper. In the economic literature, Smale [16], [17] and [18], Rand [19], Simon and Titus [20] and Wan [21] have studied local properties

of admissible points from the viewpoint of differential topology. A number of other papers (Charnes and Cooper [22], Ecker and Kouada [23], [24], Evans and Steuer [25], Gal [26], Geoffrion, Dyer and Feinberg [27], Philip [28], Sachtman [29], and Yu and Zeleny [30]) have treated algorithms for determining and investigating admissible points, primarily for linear problems.

Applications of the concept of cone dominance are varied and include efficiency in economic planning [4],[11], mathematical statistics [31], maximizing utility vectors in exchange equilibrium [4], [32], risk-return trade-offs in portfolio selection [33], [34], risk sharing and group decisions [35], and many others as suggested in [36] and the collection [37].

2. PRELIMINARIES

Throughout our discussion, we let $\mathbf{P}$ be a nonempty and nontrivial, i.e., $\mathbf{P} \neq \{0\}$, cone in $\mathbf{R}^n$. We say that an n-vector x is *preferred* to an n-vector y with respect to the cone $\mathbf{P}$, denoted $x \succ y$, when $x \neq y$ and

$$x \in y + \mathbf{P} \equiv \{y + p : p \in \mathbf{P}\}.$$

We say that a point y is *admissible* for the cone $\mathbf{P}$ over a

given set X when $y \in X$ and X contains no points preferred to y; that is,

$$X \cap (y + \mathbb{P}) = \{y\}.$$

Let $A(X)$ denote the set of all admissible points y in X.

Frequently, the cone $\mathbb{P}$ is the nonnegative orthant $\mathbb{R}^n_+$. Then $x \succ y$ when $x \geqslant y$ and $x \neq y$, and admissibility is commonly called Pareto-optimality. As a useful variant of this example, the preference cone $\mathbb{P}$ is given by

$$\mathbb{P}_k = \{0\} \cup (\lambda_1, \ldots, \lambda_k, \gamma_{k+1}, \ldots, \gamma_n) \in \mathbb{R}_n : \text{all } \lambda_i \geqslant 0 \text{ and } (\lambda_1, \ldots, \lambda_k) \neq 0\}.$$

In this case the preference relation compares only the first k components of any vector.

The preference cone $\mathbb{P}_k$ arises from the vector maximization problem of "optimizing" a vector $f(z) = (f_1(z), f_2(z), \ldots, f_k(z))$ of k real valued criteria over a subset Z of $\mathbb{R}^{n-k}$. A point $\bar{z} \in Z$ is called efficient in this problem if there is no point $z \in Z$, satisfying $f(z) \geqslant f(\bar{z})$ with the inequality strict in at least one component.

If $X = \{x = (y,z) \in \mathbb{R}^n : z \in Z \text{ and } y \leqslant f(z)\}$, then a point $\bar{x} \in X$ is admissible with respect to the cone $\mathbb{P}_k$ if and only if $\bar{x} = (\bar{y}, \bar{z})$, $\bar{y} = f(\bar{z})$ and $\bar{z}$ is efficient in the vector maximization problem. By this construction, we have expressed, and can

study, the preference order $z \succ \bar{z}$ defined by $f(z) \geqslant f(\bar{z})$, $f(z) \neq f(\bar{z})$, in terms of a cone preference $\mathbb{P}_k$ in an enlarged space.

These examples suggest that both closed and nonclosed preference cones might be studied profitably since both arise in practice.

Many properties of admissible points depend upon the separation of the sets $y + \mathbb{P}$ and X by a hyperplane. For any admissible point y, such a separation is possible whenever $\mathbb{P}$ and X are convex since $(y + \mathbb{P}) \cap X = \{y\}$. That is, there is a nonzero n-vector p^+ such that

$$p^+ \cdot x \leqslant p^+ \cdot y \leqslant p^+ \cdot (y + p) \quad \text{for all} \quad x \in X \text{ and } p \in \mathbb{P} \quad (2.1)$$

Because the right-most inequality can be restated as $p^+ \cdot p \geqslant 0$ for all $p \in \mathbb{P}$, we may reexpress (2.1) by saying that there is a nonzero vector p^+ contained in the positive polar cone $\mathbb{P}^+$ of $\mathbb{P}$ defined by

$$\mathbb{P}^+ \equiv \{p^+ \in \mathbb{R}^n : p^+ \cdot p \geqslant 0 \text{ for all } p \in \mathbb{P}\}$$

with the property that y solves the optimization problem

$$\max \{p^+ \cdot x : x \in X\}. \quad (2.2)$$

The positive polar cone is closed and convex without any assumption on $\mathbb{P}$.

There is a partial converse to this necessary condition for a point y to be admissible, which does not require any convexity assumptions. Let $\mathbf{P}_s^+$ denote the set of *strict supports* of $\mathbf{P}$ defined by

$$\mathbf{P}_s^+ \equiv \{p^+ \in \mathbf{R}^n : p^+ \cdot p > 0 \text{ for all nonzero } p \in \mathbf{P}\}.$$

If this set is nonempty we say that $\mathbf{P}$ is *strictly supported* or that $\mathbf{P}$ is a *strictly supported cone* (any $p^+ \in \mathbf{P}_s^+$ defines a supporting hyperplane $\{x \in \mathbf{R}^n : p^+ \cdot x = 0\}$ that intersects $\mathbf{P}$ only at the origin). In this terminology, the converse states that if y solves (2.2) for any strict support $p^+ \in \mathbb{P}_s^+$, then y is admissible. This fact is a simple consequence of observing that $p^+ \in \mathbf{P}_s^+$ and $x \succ y$ (i.e., $x \neq y$, $x - y \in \mathbf{P}$) implies that $p^+ \cdot (x - y) > 0$ and, therefore, that y does not solve (2.2). We distinguish these special admissible points in the following definition :

<u>Definition 2.1</u> : An admissible point $y \in A(X)$ is *strictly admissible* if it solves the maximization problem

$$\max \{p^+ \cdot x : x \in X\}$$

for some $p^+ \in \mathbf{P}_s^+$. Any other admissible point is said to be *nonstrict*.

We let $A^s(X)$ denote the set of strictly admissible points in X.

Most of our subsequent results require that preference cones be strictly supported. When $\mathbb{P}$ is closed and convex, this condition is equivalent to it being pointed, that is containing no lines. In an appendix, we extend this characterization in order to interpret the strictly supported condition directly in terms of the underlying preference cone whenever $\mathbb{P}$ is convex. The following characterization is a consequence of this development.

Proposition 2.1. *Let $\mathbb{P}$ be a convex cone in $\mathbb{R}^n$. $\mathbb{P}$ is supported strictly if and only if the relative interior of $\mathbb{P}^+$ is contained in $\mathbb{P}_s^+$.*

As an example, if $\mathbb{P} = \{p = (p_1, p_2, p_3) \in \mathbb{R}^3 : (p_1, p_2, p_3) = (0,0,0)$ or $p_1 \geqslant 0$ and $p_2 > 0\}$, then $\mathbb{P}^+ = \mathbb{P} \cap \{p \in \mathbb{R}^3 : p_3 = 0\}$, $\mathbb{P}_s^+ = \mathbb{P}^+ \setminus \{0\}$, and the relative interior of $\mathbb{P}^+$ is the set $\{p \in \mathbb{R}^3 : p_1 > 0, \ p_2 > 0 \ \text{and} \ p_3 = 0\}$, a strict subset of $\mathbb{P}_s^+$.

Remark 2.1

Throughout the remainder of this paper we frequently apply elementary results of convex analysis without reference. We also translate many properties of polar cones usually formulated in terms of the (negative) polar of any set S, denoted by $S^* \equiv \{y \in \mathbb{R}^n : y \cdot x \leqslant 0 \ \text{for all} \ x \in S\}$ into statements concerning the positive polar S^+ of S. Standard texts

in convex analysis (Rockafellar [38], Stoer and Witzgal [39]) discuss those results that we use.

In addition to the notation introduced earlier in this section, we let RC(X) denote the recession cone of a convex set, let cl(S), ri(S), and $-S \equiv \{x : -x \in S\}$ denote the closure, relative interior, and negative of any set S, and let $S \backslash T$ denote the set theoretic difference of two sets S and T. We adopt Rockafellar's [38] terminology by including the origin in RC(X), but by defining direction of recessions as only the nonzero elements of this cone. Finally, we use the Euclidean norm to define the open and closed unit balls in $\mathbf{R}^n$.

3. GLOBAL CHARACTERIZATIONS

When the set of available alternatives X is convex and closed and the preference cone $\mathbf{P}$ is convex and supported strictly, the admissible set A(X) has important global properties : it is connected (see Theorem 3.4) if some strict support to $\mathbf{P}$ makes an obtuse angle with every direction of recession of X, and it can be characterized in terms of strictly admissible points (see Theorem 3.1) if no direction of recession of X belongs to the closure of $\mathbf{P}$.

Before establishing these properties, we briefly con-
sider the existence of admissible points.

3.1 Existence

Yu [15] has observed that A(X) is nonempty if either X
is compact and the interior of $\mathbb{P}^+$ is nonempty ($\mathbb{P}^+_s \neq \phi$ suf-
fices) or if the problem $\max\{p^+ \cdot x : x \in X\}$ has a unique solu-
tion for some $p^+ \in \mathbb{P}^+$. Neither condition requires convexity
of $\mathbb{P}$ or X. He also notes that A(X) may be empty when these
conditions are not satisfied.

The following propositions characterize the existence
of admissible points for closed and convex, but not necessar-
ily bounded, sets of alternatives when preferences are defined
by strictly supported closed convex cones.

<u>Proposition 3.1</u>. *Let* $\mathbb{P}$ *be a strictly supported closed convex
cone and let* X *be a nonempty closed convex set. Then* $A(X) \neq \phi$
*if and only if the origin is the only element contained in
both* $\mathbb{P}$ *and the recession cone of* X.

<u>Proof</u> : If $y \neq 0 \in \mathbb{P} \cap RC(X)$ then $x + y \in (x + \mathbb{P}) \cap X$ for any
$x \in X$, and so no point $x \in X$ is admissible.

If $\mathbb{P} \cap RC(X) = \{0\}$, then $(y + \mathbb{P}) \cap X$ is compact for
any $y \in X$ ([38], Thm. 8.4). Consequently, for any $p^+_s \in \mathbb{P}^+_s$

there is an optimal solution z to the optimization problem $\max\{p_s^+ \cdot x : x \in (y + \mathbb{P}) \cap X\}$. The point z is strictly admissible in $(y + \mathbb{P}) \cap X$ with respect to $\mathbb{P}$. It also must be admissible in X, for if $p \neq 0 \in \mathbb{P}$ and $z + p \in X$, then $z + p \in (y + \mathbb{P}) \cap X$ is preferred to z. ■

To obtain a dual version of this proposition, we note the following theorem of the alternative.

<u>Proposition 3.2.</u> (a) *Let $\mathbb{P}$ be a strictly supported closed convex cone and let X be a closed convex set. Then exactly one of the following alternatives is valid :*

$\quad$ (I) $\quad \mathbb{P} \cap RC(X) \neq \{0\}$

$\quad$ (II) $\quad \mathbb{P}_s^+ \cap RC(X)^* \neq \phi$.

Consequently, the set of admissible points in X is nonempty if and only if $\mathbb{P}_s^+ \cap RC(X)^* \neq \phi$.

$\quad$ *(b) Moreover, if $\mathbb{P}$ is any strictly supported convex cone, not necessarily closed, then $A^s(X) \neq \phi$ implies alternative (II), and alternative (II) implies that $A^s(X) \neq \phi$ whenever X is a polyhedron.*

<u>Proof</u> : Alternatives (I) and (II) cannot both be valid since any $p \neq 0 \in \mathbb{P} \cap RC(X)$ and $p_s^+ \in \mathbb{P}_s^+ \cap RC(X^*)$ must satisfy

$p_s^+ \cdot p > 0$ and $p_s^+ \cdot p \leq 0$.[1] So suppose that $\mathbb{P}_s^+ \cap RC(X)^* = \phi$. Since $\mathbb{P}_s^+$ and $RC(X)^*$ are nonempty convex sets, they can be separated; that is, there is a nonzero vector $u \in \mathbb{R}^n$ and a real number β satisfying $u \cdot y \leq \beta \leq u \cdot p_s^+$ for all $y \in RC(X)^*$ and all $p_s^+ \in \mathbb{P}_s^+$. Since the origin is a limit point of $\mathbb{P}_s^+$ and is contained in $RC(X)^*$, $\beta = 0$. The left most inequality implies that $u \in RC(X)^{**} = RC(X)$, since the recession cone of a closed convex set is closed. The right most inequality shows that $u \cdot p_s^+ \geq 0$ for all $p_s^+ \in \mathbb{P}_s^+$ and thus $u \cdot p^+ \geq 0$ for all $p^+ \in \mathbb{P}^+$.[2] Consequently, $u \in \mathbb{P}^{++} = \mathbb{P}$ and $u \in \mathbb{P} \cap RC(X)$ so that one of alternatives (I) and (II) is always satisfied.

Since, by the previous proposition, $A(X)$ is nonempty if and only if alternative (I) is not satisfied, $A(X) \neq \phi$ if and only if alternative (II) is valid.

To prove the final assertions of the proposition, consider the optimization problem

$$\max \{p_s^+ \cdot x : x \in X\} \tag{3.1}$$

where $p_s^+ \in \mathbb{P}_s^+$. If $z \in A^s(X)$, we can select p_s^+ so that z solves this problem. Therefore $p_s^+ \cdot y \leq 0$ for all $y \in RC(X)$ implying that $p_s^+ \in RC(X)^*$ and that $\mathbb{P}_s^+ \cap RC(X)^* \neq \phi$.

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[1] Closure of $\mathbb{P}$ is not required for this conclusion.

[2] Every point of $\mathbb{P}^+$ is a limit point of $\mathbb{P}_s^+$, since, by Proposition 2.1, $\mathrm{ri}\,(\mathbb{P}^+) \subseteq \mathbb{P}_s^+ \subseteq \mathbb{P}^+$.

If $p_s^+ \in \mathbf{P}_s^+ \cap RC(X)^*$ and X is polyhedral, then $p_s^+ \cdot y \leqslant 0$ for all $y \in RC(X)$ implying, by linear programming theory, that there is a solution to problem (3.1). This solution is strictly admissible. ∎

The following examples show that the hypothesis that $\mathbf{P}$ is closed is required for these propositions and that converses to part (b) of the last proposition are not possible.

<u>Example 3.1.</u> Let $X = \{(x_1, x_2) \in \mathbf{R}^2 : x_1 x_2 \geqslant 1,\ x_1 \geqslant 0,\ x_2 \geqslant 0\}$ and let $\mathbf{P} = \{(x_1, x_2) \in \mathbf{R}^2 : x_1 < 0 \text{ and } x_2 \geqslant 0\} \cup \{(0,0)\}$.

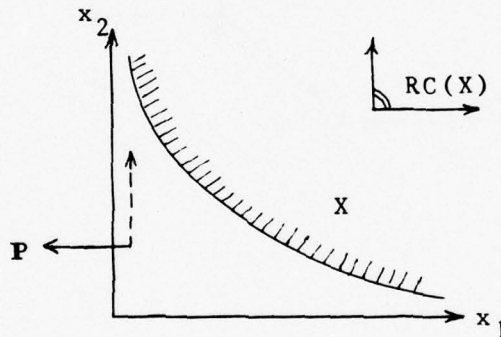Figure 3.1 $\mathbf{P} \cap RC(X) = \{0\}$ and No Admissible Points

Then $p_s^+ = (-1,0) \in \mathbf{P}_s^+ \cap RC(X)^*$ and $A(X) = \phi$. The preference cone $\mathbf{P}$ is not closed and neither of the conditions $\mathbf{P} \cap RC(X) \neq \{0\}$ or $\mathbf{P}_s^+ \cap RC(X)^* = \phi$ for $A(X)$ to be empty is valid. In addition, the converse to the first assertion in part (b) of the last proposition is violated.

Example 3.2. Let $\mathbb{P}$ be the positive orthant in R^3 except for those points in the $x_1 - x_2$ plane not on the x_2-axis, i.e.,

$$\mathbb{P} = \{(0,\lambda,0) : \lambda \geqslant 0\} \cup \{(\lambda_1,\lambda_2,\lambda_3) : \lambda_1 \geqslant 0,\ \lambda_2 \geqslant 0 \text{ and } \lambda_3 > 0\}.$$

Let $X = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_3 = 0,\ x_1 \geqslant x_2 \text{ and } x_1 \leqslant x_2 + 1\}$. Then $RC(X)$ is the line $\lambda(1,1,0),\ \lambda \in \mathbb{R}$; $RC(X)^*$ is the subspace $\{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_1 + x_2 = 0\}$; $\mathbb{P}_s^+ = \{(\lambda_1,\lambda_2,\lambda_3) \in \mathbb{R}^3 : \lambda_1 \geqslant 0,\ \lambda_2 > 0 \text{ and } \lambda_3 > 0\}$ and $A(X) = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_3 = 0 \text{ and } x_1 = x_2\}$.
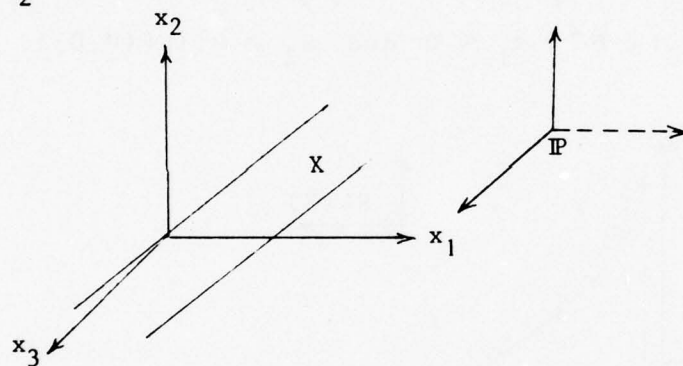

Figure 3.2   Proposition 3.2 Requires $\mathbb{P}$ to Be Closed

In this instance, $\mathbb{P}$ is not closed and neither alternative (I) or (II) of Proposition 3.2 is valid. The example also shows that the statement "$A(X) \neq \phi$ implies alternative (II)" might not be true, even when X is polyhedral, unless $\mathbb{P}$ is closed.

3.2 Representation

Arrow, Barankin and Blackwell [8] have shown that every admissible point is a limit point of strictly admissible points whenever $\mathbf{P} = \mathbf{R}_+^n$ and X is convex and compact. The following example shows that this property is not valid for all preference cones.

Example 3.3

Let X, as illustrated in Figure 3.3, be a truncated cone in $\mathbf{R}^3$ with vertex $V = (0,1,1)$ and circular base $\{x \in \mathbf{R}^3 : x_1^2 + x_2^2 \leqslant 1$ and $x_3 = 0\}$. Let $\mathbf{P} = \mathbf{P}_2 = \{0\} \cup \{(\lambda_1, \lambda_2, \lambda_3) : \lambda_1 \geqslant 0, \lambda_2 \geqslant 0$ and $(\lambda_1, \lambda_2) \neq 0\}$ which "ignores" the direction $\pm (0,0,1)$. The set of strictly admissible points $A^s(X)$ consists of those points on the interior of the arc KS, the closure of these points is the entire arc, but the admissible points also include the line segment KV
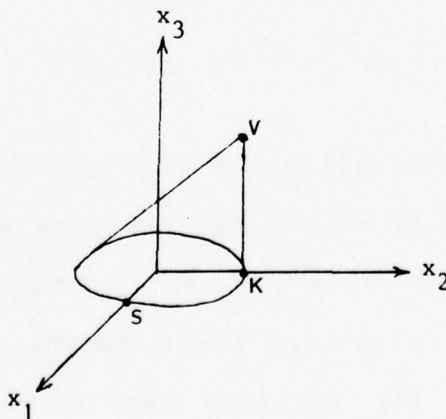

Figure 3.3 $A(X) \not\subseteq cl\ A^s(X)$

Observe that the admissible points in this example are those points in X which are translations of the arc KS by the direction (0,0,1), a direction which belongs to the closure of the preference cone $\mathbf{P}_2$ but not $\mathbf{P}_2$ itself. Our next result, which shows that this observation characterizes admissible points under very general hypothesis, extends the Arrow, Barankin and Blackwell representation by permitting more general preference cones and by relaxing compactness of of X. We build upon their clever proof techniques using the following result in place of the von Neumann minimax theorem.

<u>Lemma 3.1</u>. *Let* C *and* D *be nonempty closed convex sets in* $\mathbf{R}^n$ *satisfying the conditions*

$$(i) \quad C^* \cap RC(D) = \{0\}$$

$$(ii) \quad RC(C) \cap D^+ = \{0\}.$$

Then the function $u \cdot v$ *has a saddlepoint on* $C \times D$ *in the sense that for some* $u^o \in C$ *and* $v^o \in D$

$$u \cdot v^o \leqslant u^o \cdot v^o \leqslant u^o \cdot v \quad \text{for all } u \in C \text{ and } v \in D.$$

<u>Proof</u> : The lemma is a specialization of a theorem due to Rockafellar [38, Theorem 37.6]. By this theorem a saddlepoint exists if the functions $f_u(v) \equiv u \cdot v$ for $u \in ri(C)$ have no common direction of recession over their domain D and the functions $g_v(u) \equiv -u \cdot v$ for $v \in ri(D)$ have no common direction of recession over their domain C. A direction of recession y

for $f_u(v)$ is a nonzero element of $RC(D)$ satisfying $u \cdot y \leq 0$.
Therefore, the functions $f_u(v)$ for $u \in ri(C)$ have no common
direction of recession if no nonzero $y \in RC(D)$ satisfies
$u \cdot y \leq 0$ for all $u \in C$; that is, $RC(D) \cap C^* = \{0\}$. Similarly,
the functions $g_v(u)$ for $v \in ri(D)$ have no common direction
of recession if the inequalities $-y \cdot v \leq 0$ for all $v \in D$ are
impossible to satisfy simultaneously whenever $y \in RC(C)$ is
nonzero, which is condition (ii). ∎

Theorem 3.1. *Let* X *be a closed convex set, let* $\mathbb{P}$ *be a*
strictly supported convex cone, and suppose that
$cl \, \mathbb{P} \cap RC(X) = \{0\}$. *Then any point* $x \in A(X)$ *can be written as*

$$x = x^* - \bar{p}$$

where x^* *belongs to* $cl \, A^s(X)$ *and either* $\bar{p} = 0$ *or* $\bar{p}$ *belongs*
to $cl \, \mathbb{P} \backslash \mathbb{P}$.

Proof : We first establish preliminary results to be used in
the proof. Let B be the closed unit ball in $\mathbb{R}^n$ and let[3]

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[3]Several choices are possible. We may, in fact, choose polyhedral cones
for the S_j. Let B^o denote the open unit ball in $\mathbb{R}^n$ and for each
$j = 1, 2, \ldots$, let F_j be a finite set of points in $Q \equiv [ri(\mathbb{P}^+) \cup \{0\}] \cap B^o$
with the property that $\left(\frac{1}{j}\right)$-balls about these points cover Q. The convex
hull H_j of $\{0\} \cup F_1 \cup F_2 \cup \ldots \cup F_j$ is polyhedral and, so then, is the cone
S_j that it generates [38, Cor.19.71]. Any $x \in Q$ belongs to the interior
of a simplex in Q whose dimension equals that of Q. For some j, points
in F_j are close enough to the vertices of this simplex so that $x \in H_j$.
Therefore $\cup \{H_j : j \geq 1\} = Q$ and the union of the S_j's is $ri(\mathbb{P}^+) \cup \{0\}$.

S_j for $j = 1, 2, \ldots$, be nonempty, closed, convex and increasing cones (i.e., $S_j \subseteq S_{j+1}$) whose union is $\mathrm{ri}(\mathbf{P}^+) \cup \{0\}$. Then $\mathrm{cl}\ \mathbf{P} = \mathbf{P}^{++} = \mathrm{ri}(\mathbf{P}^+)^+ = \bigcap_{j \geqslant 1} S_j^+$. Note that for some positive integer J, $S_j^+ \cap RC(X) = \{0\}$ whenever $j \geqslant J$. Otherwise, $(B \cap S_j^+) \cap (B \cap RC(X)) \neq \{0\}$ for all j which implies by the finite intersection property [since X is closed, so is $RC(X)$] that $(B \cap \mathrm{cl}\ \mathbf{P}) \cap (B \cap RC(X)) \neq \{0\}$, contrary to hypothesis.

The sets $T_j \equiv S_j \cap B$ for $j = 1, 2, \ldots$, are nonempty, convex and compact. In addition, $T_j^+ = S_j^+$.

Now, let x^o be any admissible point in X and let $Z = \{z \in \mathbf{R}^n : z = x - x^o \text{ for some } x \in X\}$. Then Z is convex and $RC(Z) = RC(X)$. We apply Lemma 3.1 with $C = Z$, $D = T_j$ and $j \geqslant J$ where J is defined as above. Since T_j is compact, conditions (i) and (ii) of the lemma reduce to $RC(Z) \cap T_j^+ = \{0\}$ which we established previously. Therefore, for each $j \geqslant J$ there are points $t^j \in T^j$ and $z^j \in Z$ satisfying

$$z \cdot t^j \leqslant z^j \cdot t^j \leqslant z^j \cdot t \quad \text{for all } z \in Z, \ t \in T^j \qquad (3.2)$$

Since $x^o \in X$, $z = 0$ belongs to Z and

$$z^j \cdot t \geqslant z^j \cdot t^j \geqslant 0 \quad \text{for all } t \in T^j.$$

The definitions of S^j and T^j imply that the inequalities

$$z^j \cdot s \geqslant 0 \qquad \text{for all } s \in S^j \qquad (3.3)$$

are valid as well.

Because the union of the increasing sets S_j for $j = 1, 2, \ldots$ is $\mathrm{ri}\,(\mathbf{P}^+) \cup \{0\}$, the inequalities (3.3) imply that for any $p^+ \in \mathrm{ri}(\mathbf{P}^+)$, $p^+ \cdot z^j \geqslant 0$ for all j sufficiently large. Consequently, if $z^* = x^* - x^o$ is a limit point of the sequence $\{z^j\}_{j \geqslant 1}$, then $p^+ \cdot z^* \geqslant 0$ for all $p^+ \in \mathrm{ri}\,(\mathbf{P}^+)$ and z^* belongs to $\mathrm{ri}\,(\mathbf{P}^+)^+ = \mathbf{P}^{++} = \mathrm{cl}\,\mathbf{P}$. If $x^* \neq x^o$, then $(x^* - x^o) \notin \mathbf{P}$ because x^o was chosen from $A(X)$.

Therefore, any limit point z^* to the sequence $\{z^j\}_{j \geqslant 1}$ gives $x^o = x^* - z^*$ with either $z^* = 0$ or $z^* \in \mathrm{cl}\,\mathbf{P} \setminus \mathbf{P}$. This representation satisfies the conclusion of the theorem if $x^* \in \mathrm{cl}\,A^s(X)$. But x^* fulfills this condition, since, with $z^j = x^j - x^o$, expression (3.2) gives

$$t^j \cdot [x^j - x^o] \geqslant t^j \cdot [x - x^o] \quad \text{for all } x \in X.$$

Consequently, x^j solves

$$\max \{t^j \cdot x : x \in X\}$$

and $x^j \in A^s(X)$ because $t^j \in \mathbf{P}_s^+$. Thus x^* is a limit point of the strictly admissible points $\{x^j\}_{j \geqslant 1}$.

To complete the proof, we must show that the sequence $\{z^j\}_{j \geqslant 1}$ contains a limit point. Suppose, to the contrary, that it does not. Then the Euclidean norms λ^j of the z^j must grow without bound as j increases. As we have noted previously,

for each $p^+ \in ri\ (P^+)$, $p^+ \cdot z^j \geqslant 0$ and hence $p^+ \cdot \dfrac{z^j}{\lambda^j} \geqslant 0$ for all j sufficiently large. But then for any limit point y to the sequence $\left\{ \dfrac{z^j}{\lambda^j} \right\}_{j \geqslant 1}$, $p^+ \cdot y \geqslant 0$ for all $p^+ \in ri(P^+)$. Therefore $y \in ri\ (P^+) = P^{++} = cl\ P$ and $y \in RC(X) = RC(Z)$ ([38], Thm.8.2), contrary to hypothesis, and our assumption that the sequence $\left\{ z^j \right\}_{j \geqslant 1}$ contains no limit point is untenable. ∎

When the preference cone P is closed, either $P \cap RC(X) \neq \{0\}$ and $A(X) = \phi$ or $P \cap RC(X) = \{0\}$ which fulfills the hypothesis of the theorem. In either case, the characterization simplifies to :

Corollary 3.1. *Let* X *be a closed convex set and let* P *be a strictly supported closed convex cone. Then every admissible point is a limit point of strictly admissible points.*

A slight modification to Example 3.3 shows that the representation of Theorem 3.1 might not be valid when $P \cap RC(X) \neq \phi$.

Example 3.4. Let Y be the cone generated by $X - V$ in Example 3.3. Then if $P = P_2$, the admissible set of $V + Y$ is the half-line from V passing through K. The set of strictly admissible points is empty, however.

We should emphasize that the previous results do not

characterize admissible points completely. Arrow, Barankin and Blackwell show by an example that a limit point of strictly admissible points need not be admissible. The following example shows that points expressed as $x = x^* - \bar{p}$ as in Theorem 3.4 need not be admissible, even when $x^* \in cl\, A^s(X)$ is admissible.

Example 3.5   As in Example 3.2, let $\mathbb{P}$ be defined by

$$\mathbb{P} = \{(0,\lambda,0) : \lambda \geqslant 0\} \cup \{(\lambda_1,\lambda_2,\lambda_3) : \lambda_1 \geqslant 0,\ \lambda_2 \geqslant 0 \text{ and } \lambda_3 > 0\}.$$

Let X be the triangle in $\mathbb{R}^3$ given by

$$X = \{(x_1,x_2,0) : x_2 \geqslant 0,\ x_1 \geqslant x_2 \text{ and } x_1 + x_2 \leqslant 1\}.$$

Every point in X on the line segment ℓ satisfying $x_1 + x_2 = 1$ is strictly admissible and every point x in X can be written as

$$x = x^* - \bar{p}$$

for some $x^* \in \ell$ and $\bar{p} \in cl\, \mathbb{P} \setminus \mathbb{P}$; but not every point is admissible. The admissible points are the points in X on the lines $x_1 = x_2$ and $x_1 + x_2 = 1$.

3.3 Connectedness

When X is a polyhedron and $\mathbb{P}$ is a polyhedral cone, parametric analysis in linear programming shows that $A(X)$ is

connected (see, for example, Koopmans [11], or Yu and Zeleny [30] for related results). To study the connectedness of A(X) in a more general setting, we also use a parametrization of a mathematical programming objective function.

We first set some additional notation. Let $p(0)$ and $p(1)$ be any points in $\mathbf{R}^n$ and for each $0 \leq \theta \leq 1$, let $p(\theta) = \theta p(1) + (1-\theta)p(0)$. Let $X(\theta)$ denote the set of optimal solutions to the optimization problem

$$v(\theta) = \sup \{p(\theta) \cdot x : x \in X\} \qquad (3.4)$$

for a given set X, not necessarily convex; $x(\theta)$ denotes a generic element of $X(\theta)$.

Berge [40] discusses several properties of parametric optimization problems which encompass results for this problem. See also Hildenbrand and Kirman [41], whose introductory description of parametric analysis is formulated to include the following result.

Lemma 3.2. *Let X be an arbitrary subset of* $\mathbf{R}^n$ *and assume that the solution set* $X(\theta)$ *to problem (3.4) is nonempty and compact for all* $0 \leq \theta \leq 1$. *Then the point to set mapping* $\theta \rightarrow X(\theta)$ *is upper semi-continuous; that is, if* $0 \leq \theta \leq 1$ *and G is an open set in* $\mathbf{R}^n$ *containing* $X(\theta)$, *then there is a real number* $\delta > 0$ *such that* $X(\theta') \subseteq G$ *whenever* $0 \leq \theta' \leq 1$ *and* $|\theta - \theta'| \leq \delta$.

Using this lemma, we first consider the set of strictly admissible points.

Theorem 3.2. *Let* $\mathbb{P}$ *be a strictly supported convex cone, let* X *be a given subset of* $\mathbb{R}^n$, *and suppose that the solution set to the optimization problem* $\max \{p_s^+ \cdot x : x \in X\}$ *is nonempty, compact and connected for each* $p_s^+ \in \mathbb{P}_s^+$. *Then* $A^s(X)$ *is connected.*

Proof : Suppose to the contrary that $A^s(X)$ is not connected that is, there are disjoint open sets G and H with $G \cap A^s(X) \neq \phi$, $H \cap A^s(X) \neq \phi$ and $A^s(X) \subseteq G \cup H$. Let $x(0) \in G \cap A^s(X)$ and $x(1) \in H \cap A^s(X)$. The definition of strictly admissible points implies that there are vectors $p(0) \in \mathbb{P}_s^+$ and $p(1) \in \mathbb{P}_s^+$ so that for $\theta = 0$ and 1, $x(\theta)$ solves

$$\max \{p(\theta) \cdot x : x \in X\} \tag{3.5}$$

where $p(\theta) \in \mathbb{P}_s^+$ is defined as above, as $p(\theta) = \theta p(1) + (1-\theta)p(0)$. Since the set of optimal solutions $X(\theta)$ to problem (3.5) is nonempty and connected, each $X(\theta)$ is either contained in G or contained in H. In particular, $X(0) \subseteq G$ and $X(1) \subseteq H$.

Let $\theta' = \sup \{\bar{\theta} : 0 \leq \bar{\theta} \leq 1 \text{ and } X(\theta) \subseteq G \text{ for all } 0 \leq \theta \leq \bar{\theta}\}$. By the previous lemma, $\theta' > 0$. Now $X(\theta')$ must be contained in either G or H. But either assumption leads to a contradiction. For if $X(\theta') \subseteq G$, then $X(\theta) \subseteq G$ for every $\theta \leq \theta' + \delta$

and some $\delta > 0$ by Lemma 3.2, contrary to the definition of θ'; and if $X(\theta') \subseteq H$, then by Lemma 3.2 again, $X(\theta) \subseteq H$ for every $\theta \geqslant \theta' - \delta$ and some $\delta > 0$ contrary to the definition of θ'.

Therefore, our assumption that $A^s(X)$ is not connected is untenable and the theorem has been proven. ∎

The following version of this theorem is probably more useful in applications.

<u>Theorem 3.3</u>. *Let* **P** *be a strictly supported convex cone, let X be a closed convex set, and suppose that* $- \mathbf{P}_s^+ \cap RC(X)_s^+ \neq \phi$. *Then* $A^s(X)$ *is connected.*

Note that the condition $- \mathbf{P}_s^+ \cap RC(X)_s^+$ states that for some $p_s^+ \in \mathbb{P}_s^+$, $p_s^+ \cdot y < 0$ for every element y in the recession cone of X. That is, the solution set to the optimization problem $\sup \{p_s^+ \cdot x : x \in X\}$ must be bounded for some, but not necessarily all, strict supports p_s^+ to $\mathbb{P}$. This weakening of the hypothesis of Theorem 3.2 is offset by the stronger assumption that X is convex.

With slight modifications, the proof of Theorem 3.2 proves Theorem 3.3 as well. Without loss of generality, we may select $x(1)$ in the proof as the solution to $\max \{p(1) \cdot x : x \in X\}$ where $- p(1) \in RC(X)_s^+$. Then we invoke the following result instead of Lemma 3.2. (Observe that each solution set $X(\theta)$, $0 \leqslant \theta \leqslant 1$,

to problem (3.5) is convex and hence connected.)

Lemma 3.3. *Let* X *be a convex set in* $\mathbf{R}_n$. *Assume that the solution set* $X(\theta)$ *to the problem*

$$v(\theta) = \sup \{[\theta p(1) + (1 - \theta)p(0)] \cdot x : x \in X\}$$

is nonempty for $\theta = 0$ *and nonempty and compact for* $\theta = 1$. *Then* $X(\theta)$ *is nonempty and compact for all* $0 < \theta \leq 1$ *and the point to set mapping* $\theta \to X(\theta)$ *is upper semi-continuous.*

This lemma is proved in Appendix B.

When X is compact, $RC(X) = \{0\}$, every point in $\mathbf{R}^n$ is a strict support to $RC(X)$, and the hypothesis $- \mathbf{P}_s^+ \cap RC(X)_s^+ \neq \phi$ of this theorem is valid whenever $\mathbf{P}$ is strictly supported. Therefore, we have :

Corollary 3.2. *Let* $\mathbf{P}$ *be a strictly supported convex cone and let* X *be convex and compact. Then* $A^s(X)$ *is connected. If, in addition,* $\mathbf{P}$ *is closed or* $\mathbf{P} \setminus \{0\}$ *is open, then* $A(X)$ *is connected.*

Proof : As we have just noted, Theorem 3.3 shows that $A^s(X)$ is connected. Whenever $\mathbf{P} \setminus \{0\}$ is open, $\mathbf{P}^+ = \mathbf{P}_s^+$ and so $A(X) = A^s(X)$ is connected. By Corollary 3.1, whenever $\mathbf{P}$ is closed $A(X)$ is connected since it is contained in the closure of the connected set $A^s(X)$ and contains $A^s(X)$. $\blacksquare$

The next two examples show that the "strictly supported"

condition imposed upon $\mathbb{P}$ in this corollary is indispensible, as is the condition $-\mathbb{P}_s^+ \cap RC(X)_s^+ \neq \phi$ of Theorem 3.3.

Example 3.6. Let X be the closed unit ball in $\mathbb{R}^2$ and let $\mathbb{P} = \{(0,\lambda) \in \mathbb{R}^2 : \lambda \in R\}$. Then P is closed and convex, but $\mathbb{P}_s^+ = \phi$. Since the only admissible points are $(-1,0)$ and $(1,0)$, A(X) is not connected.

Example 3.7. Let $S = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_1 = 0, x_3 \leq 0$ and $e^{x_3} - 1 \leq x_2 \leq 0\}$ and let T be the halfline $\{(1,0,\lambda) \in \mathbb{R}^3 : \lambda \leq 0\}$. Define X, which is closed ([38], Cor. 9.8.1), as the convex hull of S and T and let $\mathbb{P}_s^+$ be generated by nontrivial non-negative combinations of $p(0) = (-1,0,0)$ and $p(1) = (1,-1,0)$. Since the third component of every element of $\mathbb{P}_s^+$ is zero, a point in X is strictly admissible if and only if it belongs to $y + T$ for some point y that is strictly admissible in the set obtained by projecting X onto the $x_1 - x_2$ axis.

The solutions to problem (3.5) are

$$X(\theta) = \begin{cases} S & \text{if } \theta = 0 \\ \phi & \text{if } 0 < \theta < 1 \\ T & \text{if } \theta = 1 \end{cases}$$

Alternatives X
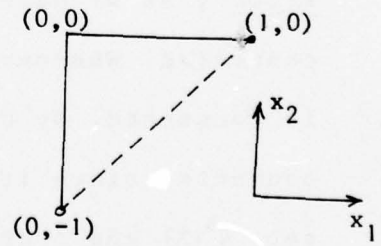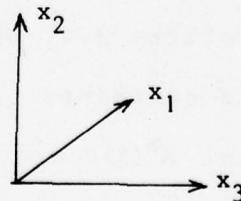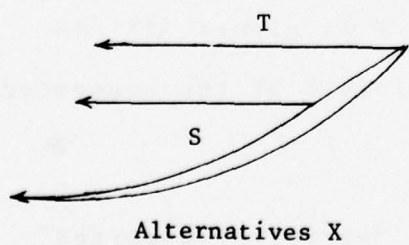
Projection on x_1-x_2 Plane

Figure 3.4

In this case, $RC(X)_s^+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 < 0\}$,
$-\mathbb{P}_s^+ \cap RC(X)_s^+ = \phi$ and $A^s(X) = S \cup T$ is not connected.

To show that $A(X)$ is connected without assuming that
the underlying preference cone $\mathbb{P}$ is closed or that $\mathbb{P} \setminus \{0\}$
is open, requires additional argument. For any vector $y \in \mathbb{R}^n$,
let $\mathcal{L}(y)$ and $\mathcal{L}(y)^\perp$ denote the subspace generated by y,
and its orthogonal subspace.

<u>Lemma 3.3</u>. *Let* $\mathbb{P}$ *and* X *be a cone and arbitrary set in* $\mathbb{R}^n$.
Suppose that p^+ *belongs to* $\mathbb{P}^+$ *and that* y *solves*
$\max \{p^+ \cdot x : x \in X\}$. *Then the following conditions are equiv-*
alent :

   (i) $y \in A(X)$

  (ii) $y \in A(\hat{X})$ *where* $\hat{X} = X \cap (y + \mathcal{L}(p^+)^\perp)$

 (iii) y *is admissible in* $X \cap (y + \mathcal{L}(p^+)^\perp)$ *with respect to*
       $\mathbb{P} \cap \mathcal{L}(p^+)^\perp$.

<u>Proof</u> : Suppose that $z = y + p \in X$ for some $p \in \mathbb{P}$. By defini-
tion of y, $p^+ \cdot z = p^+ \cdot y + p^+ \cdot p \leqslant p^+ \cdot y$. But since $p^+ \cdot p \geqslant 0$,
$p^+ \cdot p = 0$; that is, $p \in \mathcal{L}(p^+)^\perp$ and $z \in (y + \mathcal{L}(p^+)^\perp) \cap X$. Con-
sequently, y is not admissible with respect to $\mathbb{P}$, that is,
p can be chosen to be nonzero, if and only if condition (ii) is
violated and if and only if condition (iii) is violated. ∎

<u>Corollary 3.3</u>. *Let* $p^+ \in \mathbb{P}^+ \setminus \mathbb{P}_s^+$ *and let* y *solve the problem*

max $\{p^+ \cdot x : x \in X\}$. *Then either* $y \in A(X)$ *or* $y + p \in X$ *for some* p *contained in the boundary of* $\mathbf{P}$.

Proof : Apply the previous lemma, noting that any point $p \in \mathbf{P} \cap \mathcal{L}(p^+)^{\perp}$ belongs to the boundary of $\mathbf{P}$. ■

Corollary 3.4. *Let* $p^+ \in \mathbf{P}^+$, *let* $\hat{X} = \{x \in X : p^+ \cdot x \geqslant p^+ \cdot z$ *for all* $z \in X\}$ *and let* $p_s^+ \in \mathbf{P}_s^+$. *Then any solution* y *to the problem* $\max \{p_s^+ \cdot x : x \in \hat{X}\}$ *belongs to* $A(X)$.

Proof : Since y belongs to $A^s(\hat{X}) \subseteq A(\hat{X})$ and, by Lemma 3.3, $A(\hat{X}) \quad A(X) \cap \hat{X}$, y belongs to $A(X)$. ■

These results and the representation theorem 3.1 provide ingredients for proving that the set of admissible points is connected, without requiring $\mathbf{P}$ to be closed or $\mathbf{P} \setminus \{0\}$ to be open.

Theorem 3.4. *Let* $\mathbf{P}$ *be a strictly supported convex cone and let* X *be a closed convex set satisfying* $-\mathbf{P}_s^+ \cap RC(X)_s^+ \neq \phi$. *Then* $A(X)$ *is connected.*

Proof : We use induction on the dimension k of X. Whenever $k = 1$, $A(X)$ is an interval and hence connected, so assume that the theorem is valid for all closed convex sets with dimension less than k and that X has dimension k.

Note, first, that no generality is lost by assuming that

X has full dimension. For suppose, by translation if necessary, that the origin belongs to X. Then X is connected in $\mathbf{R}^n$ if and only if it is connected in L, the smallest linear subspace containing X. The definition of admissibility implies that x belongs to A(X) if and only if it is admissible in X with respect to $\mathbf{P} \cap L$. Moreover, expressing any $p_s^+ \in \mathbf{P}_s^+$, $-p_s^+ \in RC(X)_s^+$ as $p_s^+ = p_L^+ + p_{\perp}^+$ where $p_L^+ \in L$ and $p_{\perp}^+ \in L^{\perp}$, the orthogonal subspace of L, shows that $Q \equiv (\mathbf{P} \cap L)_s^+ \cap L$ is nonempty as is $-Q \cap [RC(X)_s^+ \cap L]$ (i.e., if $p \in \mathbf{P} \cap L$ and $y \in RC(X)$, then $p_s^+ \cdot p = p_L^+ \cdot p > 0$ and $p_s^+ \cdot y = p_L^+ \cdot y < 0$. Thus $p_s^+ \in Q$ and $-p_s^+ \in RC(X)_s^+ \cap L$). Consequently, the hypothesis of the theorem is valid in L and we may assume that X is an element of $\mathbf{R}^k$.

Let y be any element of A(X). Then y solves $\max \{p^+ \cdot x : x \in X\}$ for some $p^+ \in \mathbf{P}^+$. Let $\hat{X}$ denote the set of optimal solutions to this problem. If (i) $A(X) \cap \hat{X}$ is connected, and (ii) cl $A^s(X) \cap A(X) \cap \hat{X} \neq \phi$, then by Theorem 3.3 the set of strictly admissible points in X together with the admissible points in $\hat{X}$ is connected. Since $y \in A(X)$ was chosen arbitrarily, the set A(X) will be connected as well.

Consequently, we will establish the theorem by verifying conditions (i) and (ii). Since $\hat{X} \subseteq X$, $RC(\hat{X}) \subseteq RC(X)$ and $RC(X)_s^+ \subseteq RC(\hat{X})_s^+$. Therefore $-\mathbf{P}_s^+ \cap RC(\hat{X})_s^+ \neq \phi$ and, since $\hat{X}$ has dimension less than k, the inductive hypothesis implies

that $A(\hat{X})$ is connected. By Lemma 3.3, $A(X) \cap \hat{X} = A(\hat{X})$ and condition (i) is satisfied.

To establish condition (ii), let x^o solve $\max\{p_s \cdot x : x \in \hat{X}\}$ for some $p_s^+ \in \mathbb{P}_s^+$. By Corollary 3.4, $x^o \in A(X)$. Since $-\mathbb{P}_s^+ \cap RC(X)_s^+ \neq \phi$, $cl\ \mathbb{P} \cap RC(X) = \{0\}$ (any $q \in -\mathbb{P}_s^+ \cap RC(X)_s^+$ and $p \in cl\ \mathbb{P} \cap RC(X)$ would satisfy $q \cdot p \leq 0$ and $q \cdot p > 0$) and the representation theorem 3.1 applies. Thus x^o can be represented as $x^o = x^* - \bar{p}$ where $x^* \in cl\ A^s(X)$ and either $\bar{p} = 0$ or $\bar{p} \in cl\ \mathbb{P} \setminus \mathbb{P}$. Therefore $p^+ \cdot x^o = p^+ \cdot x^* - p^+ \cdot \bar{p} \leq p^+ \cdot x^*$ and $p_s^+ \cdot x^o = p_s^+ \cdot x^* - p_s^+ \cdot \bar{p} \leq p_s^+ \cdot x^*$, and so $x^* \in \hat{X}$ and x^* solves $\max\{p_s^+ \cdot x : x \in \hat{X}\}$. As a result, $x^* \in A(X) \cap cl\ A^s(X)$ and condition (ii) is satisfied. ∎

4. LOCAL CHARACTERIZATIONS

Studying properties of an underlying set by applying convex analysis to approximations of the set has been a recurring and fruitful theme in optimization. In this section we adopt this viewpoint, assuming that the set of alternatives X is defined as the intersection of a convex set C with a set D, not necessarily convex. By approximating D at a given point x^o to form an approximation to X, we investigate admissibility in X via the approximation. We show, with appropriate *hypotheses*, that strict admissibility in X is equivalent to

admissibility in the approximation. We also establish a Kuhn-Tucker theory in the setting of cone dominance, which when specialized, becomes the Kuhn-Tucker theory of nonlinear programming.

4.1 General Setting

Let us call a set $L(x^o)$ a *canonical approximation* to D at x^o if $L(x^o) - x^o \equiv \{(x - x^o) : x \in L(x^o)\}$ is a closed cone. If, in addition, $D \subseteq L(x^o)$, we say that $L(x^o)$ is a *support* to D at x^o. A support to D at x^o is said to be *finite* if it is a polyhedron. In this case, $L(x^o)$ is the intersection of a finite number of half-spaces, each supporting D at x^o.

In many applications the set D is defined by a system of nonlinear inequalities, $D = \{x : h_i(x) \geqslant 0, \ i = 1, \ldots, m\}$. In this case, two canonical approximations to D predominate in the optimization literature. When each function $h_i(x)$ is differentiable at x^o with gradient $\nabla h_i(x^o)$, then

$$L(x^o) = \{x \in \mathbb{R}^n : \nabla h_i(x^o)(x - x^o) \geqslant 0 \text{ for all i with } h_i(x^o) = 0\} \quad (4.1)$$

and when each function $h_i(x)$ is concave with supergradient s_i at x^o (i.e., s_i satisfies the "supergradient" inequality $h_i(x) \leqslant h_i(x^o) + s_i(x - x^o)$ for all $x \in \mathbb{R}^n$), then

$$L(x^o) = \{x \in \mathbb{R}^n : s_i(x - x^o) \geqslant 0 \text{ for all i with } h_i(x^o) = 0\}. \quad (4.2)$$

When the functions $h_i(x)$ are both differentiable and concave, the finite supports (4.1) and (4.2) coincide.

Our first result relates admissibility in $L(x^o)$ to strict admissibility in D. We will apply some simple, but useful, observations concerning admissible points in cones.

<u>Lemma 4.1</u>. *Let* $\mathbf{P}$ *and* Y *be closed convex cones in* $\mathbf{R}^n$ *and suppose that* $\mathbf{P}$ *is supported strictly. Then for any* $x^o \in \mathbf{R}^n$,

(i) *either* $x^o \in A(x^o + Y)$ *or* $A(x^o + Y) = \phi$,

(ii) $x^o \in A^s(x^o + Y)$ *if and only if* $\mathbf{P}_s^+ \cap Y^* \neq \phi$,

and (iii) $x^o \in A^s(x^o + Y)$ *whenever* $x^o \in A(x^o + Y)$.

<u>Proof</u> : Conclusions (i) and (ii) are immediate consequences of definitions. If $x^o \notin A^s(x^o + Y)$, then $\mathbf{P}_s^+ \cap Y^* = \phi$ by (ii) and $A(x^o + Y) = \phi$ by Proposition 3.2. This observation coupled with part (i) establishes conclusion (iii). ■

In the next two propositions we assume that $C = \mathbf{R}^n$ in the definition $X = C \cap D$ of X.

<u>Proposition 4.1</u>. *Let* $\mathbf{P}$ *be a strictly supported closed convex cone and let* $x^o \in X \subseteq \mathbf{R}^n$. *Then* $x^o \in A^s(X)$ *if and only if* x^o *is admissible in some support* $L(x^o)$ *to X at* x^o.

<u>Proof</u> : Whenever x^o is strictly admissible, it solves the problem $\max \{p_s^+ \cdot x : x \in X\}$ for some $p_s^+ \in \mathbf{P}_s^+$. The set

$L(x^o) = \{x \in \mathbb{R}^n : p_s^+ \cdot x \leqslant p_s^+ \cdot x^o\}$ supports X at x^o and x^o is admissible in $L(x^o)$.

If x^o is admissible in some support $L(x^o)$ to X at x^o, then by the previous lemma, x^o is strictly admissible in $L(x^o)$. But since $x^o \in X \subseteq L(x^o)$, the definition of strict admissibility implies that $x^o \in A^s(X)$. ∎

Examples 3.3 and 3.4 show that the closedness of $\mathbb{P}$ is necessary in the previous lemma and Proposition. Let $x^o = V = (0,1,1)$ and let X and Y be defined as in these examples. Then x^o is not strictly admissible in either $x^o + Y$ or X even though it is admissible in both of these sets and $L(x^o) = Y$ is a support to X at x^o.

Certain features of Proposition 4.1 are worth noting. First, the conclusion does not state that whenever x^o is strictly admissible it is admissible in every support $L(x^o)$ to X at this point. For example, let X be the unit cube in $\mathbb{R}^2$ and $\mathbb{P} = \mathbb{R}_+^2$. Although $L(x^o) = \{x = (x_1, x_2) : x_1 \leqslant 1\}$ is a support to X at the strictly admissible point $x^o = (1,1)$, x^o is not admissible in this support. In fact, the support $L(x^o) = \{x \in \mathbb{R}^n : p_s^+ \cdot x \leqslant p_s^+ \cdot x^o\}$ to X chosen in the proof of the proposition depends upon knowledge of a strict support $p_s^+ \in \mathbb{P}_s^+$ for which x^o maximizes $p_s^+ \cdot x$ over X. More useful would be a support that depends only upon local information

at x^o, such as the supports specified in expressions (4.1)
and (4.2). Our next results delineate a wide class of prob-
lems where such supports are possible.

Proposition 4.2. *Let* **P** *be a strictly supported closed con-*
vex cone and let X *be a polyhedron. Then every admissible*
point $x^o \in X$ *is strictly admissible. Moreover, any admissible*
point $x^o \in X$ *is strictly admissible in the support* $L(x^o)$
defined by (4.1) *with* $D = X$.

Proof : Let $h_i(x) = a^i x - b_i$ for $i = 1, 2, \ldots, m$ denote linear-
affine functions defining X and suppose that $x^o \in A(X)$ and
that $L(x^o)$ is defined by (4.1). We first note that x^o is
admissible in $L(x^o)$, for otherwise some $z \neq x^o$ belongs to
$L(x^o) \cap (x^o + \mathbf{P})$. But then $a^i \cdot z \geqslant b_i$ and $y \equiv x^o + \theta(z - x^o)$, where
$\theta \geqslant 0$, satisfies $a^i \cdot y \geqslant b_i$ for all indices i with $a^i \cdot x^o = b_i$.
Choosing θ small enough, $a^i \cdot y \geqslant b_i$ for every i with
$a^i \cdot x^o > b_i$ as well. Therefore, $y \in X$, $y \neq x^o$ and $y \in x^o + \mathbf{P}$,
contradicting $x^o \in A(X)$.

Since x^o is admissible in $L(x^o)$ and $L(x^o) - x^o$ is a
closed convex cone, $x^o \in A^o(L(x^o))$, by Lemma 4.1, and conse-
quently $x^o \in A^s(X)$. ∎

Previously, Evans and Steuer [25] have shown that
$A(X) = A^s(X)$ when **P**, as well as X, is polyhedral.

We next consider instances when X is nonpolyhedral. If x^o is strictly admissible in $X = C \cap D$, then it solves the optimization problem

$$\underset{x \in C \cap D}{\text{Max}} \ p_s^+ \cdot (x - x^o) \tag{4.3}$$

for some $p_s^+ \in \mathbf{P}_s^+$. Replacing D by $L(x^o)$, a support at x^o, and "dualizing" with respect to $u \in [L(x^o) - x^o]^+$ removes $L(x^o)$ from the constraints and gives

$$\underset{x \in C}{\text{Sup}} \ (p_s^+ + u) \cdot (x - x^o). \tag{4.4}$$

Note that when $L(x^o)$ is defined by (4.1), $L(x^o) - x^o = \{y \in \mathbf{R}^n : \nabla h_i(x^o)y \geq 0 \ \text{for all i with} \ h_i(x^o) = 0\}$ and, by Farkas' Lemma, $[L(x^o) - x^o]^+ = \{u \in \mathbf{R}^m : u = \lambda \cdot \nabla h(x)$ for some vector $\lambda \geq 0$ with $\lambda \cdot h(x^o) = 0\}$. In this case, the objective function in (4.4) is a linear approximation to the Lagrangian function of (4.3).

Recalling the usual terminology of nonlinear programming for this example, we call x^o a Kuhn-Tucker point in $X = C \cap D$ with respect to the cone $\mathbf{P}$ and support $L(x^o)$ to D at x^o, if $x^o \in D$ and

$$\underset{x \in C}{\text{Max}} \ (p_s^+ + u)(x - x^o) = 0$$

for some "Kuhn-Tucker" multipliers $p_s^+ \in \mathbf{P}_s^+$ and $u \in [L(x^o) - x^o]^+$.

The following proposition characterizes such Kuhn-Tucker points

__Proposition 4.3.__ x^o *is a Kuhn-Tucker point in X with respect to the cone* **P** *and support* $L(x^o)$ *to D at* x^o *if and only if the following two conditions are satisfied :*

(i) x^o *is strictly admissible in* $C \cap L(x^o)$ *; that is,* x^o *solves*

$$v = \max \{p_s^+ \cdot (x - x^o) : x \in C \cap L(x^o)\}$$

for some $p_s^+ \in \mathbf{P}_s^+$ *, and*

(ii) *for this* p_s^+ *,*

$$v = \min \{v(u) : u \in [L(x^o) - x^o]^+$$

where $v(u) = \sup \{(p_s^+ + u)(x - x^o) : x \in C\}$ *.*

__Proof__ : The validity of conditions (i) and (ii) implies that

$$0 = v = \max \{(p_s^+ + u)(x - x^o) : x \in C\}$$

for some $u \in [L(x^o) - x^o]^+$, so x^o is a Kuhn-Tucker point. Conversely, if x^o is a Kuhn-Tucker point with associated Kuhn-Tucker multipliers p_s^+ and u, then $v(u) = 0$. Since $u \in [L(x^o) - x^o]^+$, $u \cdot (x - x^o) \geqslant 0$ and consequently

$$p_s^+ \cdot (x - x^o) \leqslant (p_s^+ + u)(x - x^o) \leqslant v(u)$$

whenever $x \in C \cap L(x^o)$. These inequalities imply that $v \leqslant v(u) = 0$. Since $x^o \in C \cap L(x^o)$, $v \geqslant 0$. Therefore, $v = v(u) = 0$ satisfies conditions (i) and (ii). ∎

Whenever $L(x^o)$ is polyhedral, as in (4.1) or (4.2), and $ri(C) \cap L(x^o) \neq \phi$, the duality condition (ii) is fulfilled ([38], Cor. 28.2.2). In particular, if $C = \mathbf{R}^n$ then condition (ii) becomes superfluous and we may sharpen proposition 4.1 by specifying necessary conditions for strict admissibility in terms of easily computed supports.

Corollary 4.1. *Let* **P** *be a strictly supported convex cone, let* $X = D = \{x \in \mathbf{R}^n : h_i(x) \geqslant 0, \ i = 1,2,\ldots,m\}$, *and assume that each constraint function* h_i *is differentiable at a strictly admissible point* x^o *solving*

$$\max \{p_s^+ \cdot x : x \in X\} \qquad (4.5)$$

where $p_s^+ \in \mathbf{P}_s^+$. *Then, if problem (4.5) satisfies any constraint qualification[4], x^o is strictly admissible in the support* $L(x^o)$ *defined by (4.1).*

Corollary 4.2. *Let* **P** *be a strictly supported convex cone and let* $X = C \cap D$ *where* C *is a convex set and* $D = \{x \in \mathbf{R}^n : h_i(x) \geqslant 0, \ i = 1,2,\ldots,m\}$ *is defined by concave functions. Then, if* X *satisfies the Slater condition* $h_i(x^*) > 0$ *for* $i = 1,2,\ldots,m$ *for some* $x^* \in C$, x^o *is strictly admissible in the support defined by (4.2).*

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[4] Conditions, like linear independence of the vectors $\nabla h_i(x^o)$ for indices i with $h_i(x^o) = 0$, that ensure that x^o satisfies the Kuhn-Tucker conditions of nonlinear programming for problem (4.5).

4.2 The Vector Maximization Problem

To illustrate the previous results in a somewhat more concrete setting, let us consider the vector optimization problem introduced in Section 2 with a criterion function $f(z) = (f_1(z), f_2(z), \ldots, f_k(z))$ and a set of alternatives $Z \subseteq \mathbb{R}^{n-k}$. Let $X = \{(y,z) \in \mathbb{R}^n : z \in Z \text{ and } y \leqslant f(z)\}$ and suppose that Z is defined by $Z = C \cap D$ where $D = \{z \in \mathbb{R}^{n-k} : h_i(z) \geqslant 0, i = 1, 2, \ldots, m\}$. For any $z \in D$, let $h_*(z)$ denote the subvector of $h(z) = (h_1(z), h_2(z), \ldots, h_n(z))$ with components $h_i(z) = 0$. As we noted in Section 2, z^o is efficient in the vector maximization problem if and only if $x^o = (f(z^o), z^o)$ is admissible in X with respect to the preference cone $\mathbb{P} = \mathbb{P}_k$. Note, in this cone, that $\mathbb{P}_s^+ = \{(\lambda, \gamma) \in \mathbb{R}^k \times \mathbb{R}^{n-k} : \lambda > 0 \text{ and } \gamma = 0\}$.

If each of the functions $f_i(z)$ and $h_j(z)$ is differentiable, then the linear approximation to X at x^o corresponding to (4.1) becomes

$$L(x^o) - x^o = \{(y,z) \in \mathbb{R}^n : y \leqslant \nabla f(z^o) \cdot z \text{ and } \nabla h_*(z^o) \cdot z \geqslant 0\}.$$

Since any admissible point (y^*, z^*) in $L(x^o) \cap C$ with respect to $\mathbb{P}_k$ must satisfy $y^* = \nabla f(z^o) \cdot z^*$, $x^o = (y^o, z^o)$ is strictly admissible in $L(x^o) \cap C$ whenever it solves

$$v = \max\{\lambda \nabla f(z^o) \cdot (z - z^o) : z \in C \text{ and } \nabla h_*(z^o) \cdot (z - z^o) \geqslant 0\} \quad (4.6)$$

for some positive k-vector λ.

As we have noted previously, Farkas' Lemma implies that the polar to the polyhedral cone $L(x^o) - x^o$ is given by

$$[L(x^o) - x^o]^+ = \{(u^1, u^2) \in \mathbf{R}^n : u^1 = -\lambda \text{ and } u^2 = \lambda \nabla f(z) + \mu \nabla h_*(z)$$

$$\text{for some } \lambda \geqslant 0 \text{ and } \mu \geqslant 0\}.$$

Therefore, (y^o, z^o) is a Kuhn-Tucker point if it solves

$$\max_{\substack{y \in \mathbf{R}^k \\ z \in C}} \{[\lambda \nabla f(z^o) + \mu \nabla h(z^o)] \cdot (z - z^o) + (\sigma - \lambda) \cdot (y - y^o)\}$$

for some positive k-vector σ. The value of this optimization problem is $+\infty$ unless $\sigma = \lambda$. Thus, $(f(z^o), z^o)$ is a Kuhn-Tucker point, or z^o is a Kuhn-Tucker point in the vector maximization problem, whenever

$$\max_{z \in C} \{[\lambda \nabla f(z^o) + \mu \nabla h_*(z^o)] \cdot (z - z^o)\} = 0 \qquad (4.7)$$

for some positive k-vector λ.

Note that for z^o to be a Kuhn-Tucker point to the vector maximization problem requires a choice of positive weights λ for the vector criterion function $f(z)$ so that z^o is a Kuhn-Tucker point to the nonlinear program

$$\max \{\lambda f(z) : z \in C \text{ and } h(z) \geqslant 0\}. \qquad (4.8)$$

Proposition 4.3 shows that necessary and sufficient conditions

for z^o to be a Kuhn-Tucker point are that (i) z^o solves the first order linear approximation (4.6) to (4.8) at z^o, and (ii) the inequality constraints of (4.6) can be incorporated within the objective function by an appropriate choice of weights μ so that the optimal value to the problem remains unaltered. That is, Kuhn-Tucker points are associated with (i) a *regularity condition* guaranteeing that a linear approximation inherits certain solutions from a nonlinear program, as well as (ii) a *duality condition* guaranteeing dualization of the linear approximation problem.

When $k = 1$ the vector maximization problem becomes a nonlinear program and condition (4.8), with the positive scalar λ normalized to value 1, reduces to the usual Kuhn-Tucker conditions. Consequently, the regularity and duality conditions for characterizing Kuhn-Tucker points subsume all of the numerous constraint qualification conditions of nonlinear programming (see, for example, Mangasarian [42]). Fiacco and McCormick [43] seem to have first stated this fact when $C = \mathbb{R}^n$ and the duality condition is not required. In a section of an unpublished report, Magnanti [44] introduced the duality condition in the context of nonlinear programming. Halkin [45] presents related results in the context of nonlinear programming. More recently, Robinson [46] has studied optimality conditions for preference orderings in infinite dimensional spaces.

When specialized to the vector maximization problem, Corollary 4.1 shows that if $C = \mathbb{R}^n$ and the regularity condition is fulfilled, then a necessary condition for z^o to be strictly admissible (i.e., z^o solves problem (4.8) or, equivalently $(f(z^o), z^o)$ is strictly admissible in X) is that $x^o = (f(z^o), z^o)$ is admissible in $L(x^o)$. The last condition is equivalent to z^o being efficient in $Z^L \equiv \{z \in \mathbb{R}^{n-k} : \nabla h_*(z^o) \cdot (z - z^o) \geq 0\}$ with respect to the vector criterion $\nabla f(z^o) \cdot z$. Therefore efficiency in the linear approximation to the vector optimization problem is necessary for strict admissibility in the problem itself.

Remark 4.1. These results are related to the notion of proper efficiency introduced by Geoffrion [12] (see also Kuhn and Tucker [10]). By definition, z^o is a proper efficient point in the vector maximization problem if there is a scalar $M > 0$ with the property that for every $z \in Z$ and each index i satisfying $f_i(z) > f_i(z^o)$ the inequality

$$\frac{f_i(z) - f_i(z^o)}{f_j(z^o) - f_j(z)} \leq M$$

is valid for some index j such that $f_j(z) < f_j(z^o)$. As Geoffrion shows, whenever Z is a convex set and each function $f_j(z)$ for $j = 1, 2, \ldots, k$ is concave, proper efficiency of z^o is equivalent to z^o solving problem (4.8) for some $\lambda > 0$.

As we have noted, this last condition is equivalent to $(f(z^o), z^o)$ being strictly admissible in X. If $C = \mathbf{R}^n$, we may, then, restate our comment made just prior to this remark as : if z^o is strictly admissible in the vector maximization problem and the regularity condition is fulfilled, then

$z^o \in Z$ is a proper efficient point in Z^L with respect to the vector criterion $\nabla f(z^o) \cdot z$. (4.9)

We might also note that if $f(.)$ and $h(.)$ are concave, then condition (4.9) implies that z^o is a proper efficient point in the vector maximization problem. To establish this fact, we note that condition (4.9) implies that z^o solves problem (4.6) for some positive k-vector λ. Therefore z^o satisfies the Kuhn-Tucker conditions (4.7) with $C \in \mathbf{R}^n$ and, because of our concavity hypothesis, z^o solves the Lagrangean maximization

$$\max_{z \in \mathbf{R}^{n-k}} \{\lambda f(z) + \mu h_*(z)\}.$$

Since, by definition, $\mu h_*(z^o) = 0$, the optimal value to this problem equals $\lambda f(z^o)$, and since $z^o \in Z$, it must solve[5] the optimization problem $\max \{\lambda f(z) : h(z) \geqslant 0\}$ and hence be a proper efficient point in the vector maximization problem.

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[5]Here we use the standard weak duality argument of nonlinear programming.

We should point-out, though, that a proper efficient point z^O need not satisfy condition (4.9). As an example, let $z^O = (0,0)$ in the vector maximization problem with criterion $f_1(z) = z_1$, $f_2(z) = z_2$ and constraints $h_1(z) = z_2 - z_1^2 \geqslant 0$ and $h_2(z) = - z_2 - z_1^2 \geqslant 0$. In this instance, the regularity condition fails since the origin is not admissible in $Z^L = \{(z_1, z_2) \in \mathbf{R}^2 : z_2 = 0\}$.

5. DISCUSSION

In the previous sections we have studied structural properties of admissible points with respect to a convex cone. Our results provide global characterizations of admissible points in terms of strictly admissible points and local characterizations in terms of linear approximations. We have also shown, with appropriate hypotheses imposed upon the problem structure, that the sets of admissible and strictly admissible points are both connected. In this section, we briefly discuss a few potential extensions and applications.

First, we might comment on the frequently evoked assumption that the underlying preference cone is strictly supported. According to Proposition A.1, this assumption rules out

"grass is greener" preferences in which each of two alternatives is preferred to the other. More generally, it does not permit situations in which $x \succ y$ and $y^j \succ x$ for $j = 1,2,\ldots$ for some points y^j converging to y. As an example, lexicographic orderings define preference cones that are not strictly supported.

Whenever underlying preferences are described by a closed indifference cone $\mathbf{P}^I$ (i.e., $y \in x + \mathbf{P}^I$ if and only if $y \succcurlyeq x$), the set $\mathbf{P} = \{x \in \mathbf{P}^I : 0 \succcurlyeq x\}$ describes a strictly supported preference cone. This cone is strictly supported for if $p \in \mathbf{P}$ belongs the lineality space of cl $\mathbf{P}$, i.e., $-p \in \mathrm{cl}\ \mathbf{P} \in \mathrm{cl}\ \mathbf{P}^I = \mathbf{P}^I$, then $0 = p - p \in p + \mathbf{P}^I$ or $0 \succcurlyeq p$, a contradiction. We should emphasize, however, that even though this construction provides strictly supported cones, our development does not presume the existence of any "weak" preference relation $\succcurlyeq$.

There are several ways in which our results might be extended. Replacing the preference cone $\mathbf{P}$ by a convex set C or, more generally, by a family of convex sets C_x, C_x denoting the set of points preferred to x, would add possibilities for broader applications. Another line of investigation would be to retain our hypothesis and to see what additional assumptions might lead to stronger conclusions. For example, Arrow and Hahn [4] show that if $\mathbf{P} = \mathbf{R}_n^+$ then the following restrictions on the (convex) set of alternatives X

(i) 0 belongs to the interior of X;

(ii) $X \cap \mathbf{R}_n^+$ is compact; and

(iii) free disposal, i.e., $x - y \in X$ for any $y \in \mathbf{R}_n^+$

whenever $x \in X$

imply that $A(X)$ is homeomorphic to an $(n-1)$-dimensional simplex. This substantial strengthening of connectedness is possible, with similar hypothesis, for other closed convex cones as well. In general, it may be that the set $A(X)$ is homeomorphic to a union of simplices with some special structure.

In a paper not yet available to us, Naccache [47] has initiated investigations of another nature. He studies stability of the set of admissible points with respect to perturbations in X and $\mathbf{P}$. He has also studied connectedness of $A(X)$, but with assumptions that may be related to free disposal.

There are a number of ways in which the structural properties discussed in this paper and these extensions aid decision making. Consider, for example, the vector optimization problem. In practice, it is convenient to generate strictly admissible points by solving

$$\max \left\{ \sum_{j=1}^{k} \lambda_j f_j(z) : z \in Z \right\} \qquad (5.1)$$

with positive weights λ_j associated with the criterion functions. By varying the weights, decision makers can generate all strictly admissible points, or by choosing a sequence of positive weights appropriately they might move toward some admissible point that is "best" with respect to some auxiliary criterion (see, for example, [27]). Considering, as before, the vector maximization problem in terms of the set $X = \{(y,z) \in \mathbb{R}^n : y \leqslant f(z) \text{ and } z \in Z\}$ and the preference cone $\mathbb{P} = \mathbb{P}_k$, we see from the representation theorem that every admissible point $(y^o = f(z^o), z^o)$ is a limit point of solutions $(y^* = f(z^*), z^*)$ to (5.1) or a translation of such limit points by vectors $(0,z) \in \text{cl } \mathbb{P}_k \setminus \mathbb{P}_k$. That is, "in the space $\mathbb{R}^k$ the image $f(.)$ of efficient points is contained in the closure of the image of the proper efficient points" (see Geoffrion [12])[6]. In this context, the representation theorem 3.1 shows that the solutions to (5.1) delineate all potential values of the criterion function when evaluated at efficient points; connectedness of the admissible points shows that to move from any (proper) efficient point to another, we can restrict ourselves to local movements among

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[6] Some assumption such as the hypothesis cl $\mathbb{P} \cap RC(X)$ of Theorem 3.1 is required for this statement. For example, let V and Y be defined as in example 3.4, let $Z = V + Y$ and let $f_1(z) = z_1$ and $f_2(z) = z_2$. The efficient set is the halfline from V passing through the point K (see Figure 3.1). Every efficient points is non-proper, though, so that the statement is not valid in this instance.

(proper) efficient points only, such as local changes in the coefficients λ_j of (5.1).

We would expect similar benefits from the structural properties of admissible points in general, especially when solving for strictly admissible points is attractive computationally.

One applications of admissibility that might be explored profitably concerns the optimization of monotonic functions $h(x)$, where, say, $h(x)$ is strictly increasing in each component x_j of the vector x. In this instance, any optimal solution to the problem

$$\max \{h(x) : x \in X\}$$

is admissible for X with respect to $\mathbf{P} \in \mathbf{R}_+^n$. This observation suggests that optimization algorithms might restrict their search to the admissible points $A(X)$, especially once any algorithm first identifies a point in this set.

Do mathematical programming algorithms have this property ? The answer, at least in terms of the simplex method, is no. In the example,

$$\max \quad 2x_1 + x_2$$
$$\text{subject to} \quad 4x_1 + 7x_2 \geqslant 18$$
$$x_1 + 2x_2 \leqslant 5$$
$$x_1 \geqslant 0, x_2 \geqslant 0$$

the admissible set is the line segment joining the points $a = (1,2)$ and the optimal solution $c = (5,0)$. Starting from the extreme point a, the simplex method moves off the admissible set to the point $b = (4.5,0)$. In a number of experiments conducted on larger problems [36], we have never observed this same phenomenon. In these examples, once the simplex method first encountered an admissible point, it always generated an admissible point at each successive iteration. Our understanding of the simplex method might be enhanced if we would explain this behavior.

6. APPENDIX A

The following, rather intuitive, propositions characterize the strictly supported condition for a convex cone. Recall that the lineality space L of any cone C is the set of lines contained in C, i.e., $L = C \cap (-C)$.

Proposition A.1. *Let* L *be the lineality space for the closure of the convex cone* **P**. *Then* **P** *is supported strictly if and only if* $\mathbf{P} \cap L = \{0\}$.

Proof : Let $L^{\perp}$ denote the orthogonally complementary subspace to L. Then $cl\ \mathbf{P} = L \oplus (cl\ \mathbf{P} \cap L^{\perp})$ is a direct sum representa-

tion, and $(\mathrm{cl}\ \mathbf{P})^+ = L^\perp \cap (\mathrm{cl}\ \mathbf{P} \cap L^\perp)^+$.

Now $p_s^+ \in L^\perp$ whenever $p_s^+ \in \mathbf{P}_s^+ \subseteq (\mathrm{cl}\ \mathbf{P})^+$ and $p_s^+ \cdot p = 0$ for all $p \in \mathbf{P} \cap L$. Consequently, $\mathbf{P} \cap L = \{0\}$.

To establish the converse, note that since $\mathrm{cl}\ \mathbf{P} \cap L^\perp$ is convex, closed and pointed, its positive polar $(\mathrm{cl}\ \mathbf{P} \cap L^\perp)^+$ has full dimension ([38], Cor.14.6.1). Any point y belonging to the interior of $(\mathrm{cl}\ \mathbf{P} \cap L^\perp)^+$ must satisfy $y \cdot p > 0$ for all nonzero $p \in (\mathrm{cl}\ \mathbf{P} \cap L^\perp)$. Expressing y as $y = y_L + y^\perp$ with $y_L \in L$ and $y^\perp \in L^\perp$, we note that $(y_L + y^\perp) \cdot p = y^\perp \cdot p > 0$ for all nonzero $p \in (\mathrm{cl}\ \mathbf{P} \cap L^\perp)$. If $p \in \mathbf{P}$ and $\mathbf{P} \cap L = \{0\}$, then $p = p_L + p^\perp$ for some $p_L \in L$ and some nonzero $p^\perp \in (\mathrm{cl}\ \mathbf{P} \cap L^\perp)$, and $y^\perp \cdot p = y^\perp \cdot p^\perp > 0$. Therefore $\mathbf{P} \cap L = \{0\}$ implies that $y^\perp \cdot p > 0$ for all $p \in \mathbf{P}$; that is, $\mathbf{P}$ is supported strictly.

∎

The proof of this proposition shows that whenever $\mathbf{P} \cap L = \{0\}$, any point contained in both $L^\perp$ and the interior of $(\mathrm{cl}\ \mathbf{P} \cap L^\perp)^+$ is a strict support to $\mathbf{P}$. We next establish the converse to this statement.

<u>Proposition A.2</u>. *Let L be the lineality space for the closure of the convex cone* $\mathbf{P}$. *Then* $\mathrm{ri}(\mathbf{P}^+) \subseteq \mathbf{P}_s^+$ *if and only if* $L \cap \mathbf{P} = \{0\}$. *Moreover, if* $\mathrm{cl}\ \mathbf{P} \cap L^\perp = \mathbf{P} \cap L^\perp$, *then* $\mathbf{P}_s^+ \subseteq \mathrm{ri}(\mathbf{P}^+)$ *and, consequently,* $\mathrm{ri}(\mathbf{P}^+) = \mathbf{P}_s^+$ *if and only if* $L \cap \mathbf{P} = \{0\}$.

<u>Proof</u> : Let $\text{cl } \mathbf{P} = L \oplus (\text{cl } \mathbf{P} \cap L^{\perp})$ be a direct sum representation. Then $\mathbf{P}^+ = (\text{cl } \mathbf{P})^+ = L^{\perp} \cap (\text{cl } \mathbf{P} \cap L^{\perp})^+$ and, since $L^{\perp}$ is a subspace and $(\text{cl } \mathbf{P} \cap L^{\perp})^+$ has a nonempty interior I which intersects $L^{\perp}$, $\text{ri }(\mathbf{P}^+) = L^{\perp} \cap I$.

By the remark preceding the proposition, $\mathbf{P} \cap L = \{0\}$ implies that $\text{ri}(\mathbf{P}^+) \subseteq \mathbf{P}_s^+$. Conversely, if $\text{ri}(\mathbf{P}^+) \subseteq \mathbf{P}_s^+$, then $\mathbf{P}_s^+ \neq \phi$ and $L \cap \mathbf{P} = \{0\}$ by the previous proposition.

Finally, suppose that $\text{cl } \mathbf{P} \cap L^{\perp} = \mathbf{P} \cap L^{\perp}$ and let y belong to the relative boundary of $\mathbf{P}^+$. Then there are vectors $y^j \in L^{\perp}$ converging to y satisfying $y^j \cdot p^j < 0$ for some $p^j \in \mathbf{P}$ which we scale to unit norm. Any limit point p of the sequence $\{p^j\}_{j \geqslant 1}$ belongs to $\text{cl } \mathbf{P} \cap L^{\perp} = \mathbf{P} \cap L^{\perp} \subseteq \mathbf{P}$ and satisfies $y \cdot p \leqslant 0$. Consequently, $y \notin \mathbf{P}_s^+$ and $\mathbf{P}_s^+ \subseteq \text{ri}(\mathbf{P}^+)$. ∎

When combined, these propositions establish Proposition 2.1 of the text. Note that the illustration following Proposition 2.1 shows that $\text{ri}(\mathbf{P}^+) \neq \mathbf{P}_s^+$ is possible.

7. APPENDIX B

We prove the continuity lemma required for Theorem 3.3, namely

Lemma 3.3. *Let X be a convex set in* $\mathbf{R}^n$. *Assume that the solution set to the problem*

$$v(\theta) = \sup \{[\theta p(1) + (1 - \theta)p(0)] \cdot x : x \in X\}$$

is nonempty for $\theta = 0$ *and nonempty and compact for* $\theta = 1$. *Then* $X(\theta)$ *is nonempty and compact for all* $0 < \theta \leqslant 1$ *and the point to set mapping* $\theta \to X(\theta)$ *is upper semi-continuous.*

Proof : Let y be any recession direction of X. Since X(0) is nonempty and X(1) is nonempty and compact, $p(0) \cdot y \leqslant 0$ and $p(1) \cdot y < 0$. Therefore

$$[\theta p(1) + (1 - \theta)p(0)] \cdot y < 0$$

for all $0 < \theta \leqslant 1$ implying that $X(\theta)$ is nonempty and compact.

According to Lemma 3.2, the mapping $\theta \to X(\theta)$ is upper semi-continuous on the interval $[\varepsilon, 1]$ for any $0 < \varepsilon \leqslant 1$. Consequently, to complete the proof we must show that this mapping is upper semi-continuous at $\theta = 0$.

This indeed is the case if, for some $\delta > 0$, the set $S_\delta \equiv \cup \{X(\theta) : 0 < \theta \leqslant \delta\}$ is bounded. For if the mapping $\theta \to X(\theta)$ is not upper semi-continuous at $\theta = 0$, then there is an open

set G containing $X(0)$ and points $x^j \in X(\theta^j) \setminus G$ for some real numbers $\theta^j > 0$ approaching 0. Since

$$p(\theta^j) \cdot x^j \geqslant p(\theta^j) \cdot x \quad \text{for all } x \in X$$

any limit point x^* (such a limit point exists since the x^j eventually lie in the bounded set S_δ) to the sequence $\{x^j\}_{j \geqslant 1}$ satisfies $x^* \notin G$ and

$$p(0) \cdot x^* \geqslant p(0) \cdot x \quad \text{for all } x \in X.$$

But then $x^* \in X(0) \setminus G$, contradicting $X(0) \subseteq G$.

Therefore to establish the theorem we only need to show that S_δ is bounded for some $\delta > 0$. For notational simplicity suppose, by translation if necessary, that $0 \in X(0)$. Then by definition

$$p(0) \cdot x(\theta) \leqslant p(0) \cdot x(0) = 0 \tag{1}$$

for any $x(\theta) \in X(\theta)$, $0 < \theta \leqslant 1$. Since $0 \in X$,

$$[\theta p(1) + (1-\theta)p(0)] \cdot x(\theta) \geqslant 0$$

implying, from (1), that

$$p(1) \cdot x(\theta) \geqslant 0. \tag{2}$$

Now if S_δ is unbounded for every $0 < \delta \leqslant 1$, then there are $\theta^j \to 0$ and points $x(\theta^j) \in X(\theta^j)$ whose Euclidean norms λ_j approach $+\infty$. Since $0 \in X$, any limit point y to the sequence $\{x(\theta^j) / \lambda_j\}_{j \geqslant 1}$ is a direction of recession of X. But the inequality (2) implies that $p(1) \cdot y \geqslant 0$, contradicting the

hypothesis that $X(1)$ is bounded. Consequently, S_δ must be bounded for some $\delta > 0$ and the point is complete. ∎

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We study the set of admissible (pareto-optimal) points of a closed convex set X when *is studied*
preferences are described by a convex, but not necessarily closed, cone. Assuming that
the preference cone is strictly supported and making mild assumptions about the recession
directions of X, *is extending* a representation theorem of Arrow, Barankin and
Blackwell by showing that all admissible points are either limit points of certain
"strictly admissible" alternatives or translations of such limit points by rays in
the closure of the preference cone, and show that the set of strictly admissible

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20. ABSTRACT CONTINUED

cont.

points is connected, as is the full set of admissible points. Relaxing the convexity assumption imposed upon X, ~~we also consider~~ local properties of admissible points, in terms of Kuhn-Tucker type characterizations. ~~We specify~~ necessary and sufficient conditions for an element of X to be a Kuhn-Tucker point, conditions which, in addition , provide local characterizations of strictly admissible points.